

# Nonlinear viscoelastic membranes

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## Abstract

The subject of nonlinear viscoelastic membranes is an important class of problems within nonlinear viscoelasticity that involves interesting and important applications, computational issues and applied mathematics.

The viscoelastic materials discussed in this paper are described by nonlinear single integral constitutive equations. After presenting the general constitutive framework, two fundamental membrane problems are formulated: the inflation of a circular membrane and the extension and inflation of a circular tube. Both problems involve large axially symmetric deformations and lead to a system of nonlinear partial differential–Volterra integral equations. A numerical method of solution is presented that combines methods for solving nonlinear Volterra integral equations and nonlinear ordinary differential equations. Finally, some properties of the equations are discussed that are related to the possibility that there may exist a critical time when the solution develops multiple branches.

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## 1. Introduction

The subject of nonlinear elastic membranes has played a prominent role in the development and application of nonlinear elasticity. First, it has been used in the determination and evaluation of the form of strain energy density function for isotropic, incompressible elastic materials. In this regard, the problem of the inflation of a circular membrane by lateral pressure has been of particular importance. It describes an experiment that produces a readily measurable axisymmetric non-homogenous deformation. Treloar [1] carried out such an experiment and measured the deformed profiles and stretch ratio distributions in a vulcanized latex rubber membrane at different levels of inflation. Adkins and Rivlin [2] used the theory of nonlinear elastic membranes to calculate the inflated profiles for several forms of the strain energy function and compared the results with the data provided by Treloar. Studies with other forms of the strain energy function were carried out by Klingbeil and Shield [3] and Hart-Smith and Crisp [4]. Wineman, Wilson and Melvin [5] showed how the measured profiles and stretch ratio distributions can be used as part of a material identification method to determine the form of the strain energy function. Recently, the inflation of a circular membrane was used to determine properties of elastomers by Przybylo and Arruda [6] and of several polymeric materials by Li, Nemes and Derdouri [7].

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Nonlinear elastic membranes have been an important area of application of nonlinear elasticity. A complete listing of the membrane shapes and loadings that have been studied is beyond the scope of this paper, but it is useful to present a representative sampling. Kydonieffs and Spencer [8] treated the finite inflation of an elastic toroidal membrane of circular cross section and also the inflation of a tube sealed at both ends into an axisymmetric surface, Wu [9] studied the contact problem of inflated cylindrical membranes with a life raft as an example while Yang and Hsu [10] studied the axisymmetric indentation of a circular membrane by a spherical indenter. In these examples, the membrane material is isotropic and incompressible and the inflation process is isothermal. However, theories have been recently developed for membranes with entropic thermoelasticity (Taylor and Steigmann [11]) and for magnetoelastic membranes (Steigmann [12]).

A third reason for the interest in nonlinear elastic membranes is that they can exhibit interesting behavior such as non-uniqueness and instability. Beatty [13] discussed some of these features for the inflation of a spherical balloon. The same behavior has also been shown to occur during the inflation of a circular sheet (Yang and Feng [14]) and an elastic toroidal membrane of circular cross section (Kydonieffs and Spencer [8]). The occurrence of these phenomena in membranes has been used by Humphrey and Canham [15] in a study of the mechanics of intercranial saccular aneurysms.

Nonlinear elasticity in general and nonlinear elastic membranes in particular have reached reasonably advanced states of development because the constitutive equation has a well-defined mathematical structure in terms of a single material property, the strain energy density function. Nonlinear viscoelasticity, on the other hand, is not as well developed. A major reason is that the mathematical structure of the constitutive equation is less well-defined. There are several candidate constitutive equations in the literature, each giving the stress in terms of a nonlinear Volterra integral operator on the deformation history. Each has different mathematical forms for this dependence and different material properties. There is another reason why nonlinear viscoelasticity is less developed than nonlinear elasticity. The response is time dependent and involves the solution of Volterra integral equations, a subject that is less familiar than differential equations to most researchers in mechanics.

There are a number of significant applications of nonlinear viscoelasticity. Elastomeric materials are slightly viscoelastic, but this can be ignored for many purposes. However, the determination of the heat generated by mechanical working requires the use of nonlinear viscoelasticity. Drape and vacuum forming are manufacturing processes that are used to deform a polymer sheet into some other shape. The process time and the material response time influence the accurate determination of the final shape. Nonlinear viscoelasticity provides the correct framework for simulating such processes. Nonlinear elasticity has often been used to model the mechanical response of soft biological tissue. However, biological tissues exhibit nonlinear viscoelasticity and certain studies require its inclusion in an appropriate model.

The purpose of this article is to show that nonlinear viscoelastic membranes can contribute to the development of nonlinear viscoelasticity just as nonlinear elastic membranes contributed to the development of nonlinear elasticity. The inflation of a circular viscoelastic membrane can be used to evaluate the predictive capability of a proposed constitutive equation. Each of the nonlinear elastic membrane applications mentioned above has a viscoelastic counterpart that can be used to study the response to time dependent loading. Also, the time dependent inflation of viscoelastic membranes can exhibit phenomena such as the branching of solutions or a jump discontinuity at some critical time. Although the list is not extensive, there have been applications of nonlinear viscoelasticity. A number of workers have studied the inflation of a planar circular membrane. Wineman [16] considered a membrane of styrene-butadiene rubber. Feng [17] later carried out a study for a latex rubber membrane. Hasseger, Kristensen, Larsen and Neergaard [18] studied the inflation and instability of a polymeric fluid membrane. Wineman also studied the extension and inflation of a tubular membrane of a polymeric fluid [19]. In summary, the subject of nonlinear viscoelastic membranes generates many interesting physical and mathematical issues.

Attention is restricted here to membranes composed of isotropic incompressible viscoelastic materials that are undergoing axisymmetric deformations. For the applications considered here, the viscoelastic material can be a solid or a fluid. These can be described by nonlinear single integral constitutive equations. General forms for these constitutive equations are introduced in Section 2. They contain most of the models in the literature as special cases. The problem of the inflation of an initially planar circular membrane is formulated in Section 3. It is shown that the problem can be reduced to the solution of a system of nonlinear partial differential–Volterra integral equations. The problem of the inflation and extension of a cylindrical tubular membrane is formulated in Section 4. This too can be reduced to the solution of a system of nonlinear partial differential–Volterra integral equations. Section 5 outlines

a method for the numerical solution of the circular and tubular membrane problems. It is readily extended to other axisymmetric membrane problems. Section 6 uses the inflation of a spherical membrane to explore the existence of a critical time at which the solution can become discontinuous. Section 7 contains final comments.

## 2. Nonlinear single integral constitutive equations

The mechanical response of viscoelastic materials depends on the history of deformation and they are thus described as materials with memory. Let ‘ $s$ ’ denote a generic time during such a deformation history that varies between the initial time  $s = 0$  and the current time  $s = t$ . A body of a viscoelastic material will pass through a sequence of configurations as ‘ $s$ ’ increases from  $s = 0$  to  $s = t$ . Let  $\mathbf{X}$  denote the position of a material particle of the body in its configuration at  $s = 0$ ,  $\mathbf{x}(s)$  be its position in the configuration at time ‘ $s$ ’ and  $\mathbf{x}(t)$  be its position in the current configuration. The motion of the body is described by specifying the relation  $\mathbf{x}(s) = \chi(\mathbf{X}, s)$  for all times  $s$  of interest and all  $\mathbf{X}$  in the initial configuration. For brevity, the current time is henceforth omitted from the notation for variables associated with the current time, i.e.  $\mathbf{x}(t) = \mathbf{x}$ .

The deformation gradient of the configuration at time ‘ $s$ ’ with respect to the configuration at  $s = 0$  is  $\mathbf{F}(s) = \partial \mathbf{x}(s) / \partial \mathbf{X}$ . The corresponding right Cauchy–Green tensor is  $\mathbf{C}(s) = \mathbf{F}(s)^T \mathbf{F}(s)$ . The deformation gradient of the current configuration with respect to the configuration at  $s = 0$  is  $\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X}$  and the corresponding left Cauchy–Green tensor is  $\mathbf{B} = \mathbf{F} \mathbf{F}^T$ . The relative deformation gradient of the configuration at time ‘ $s$ ’ with respect to the current configuration is  $\mathbf{F}_t(s) = \partial \mathbf{x}(s) / \partial \mathbf{x}(t)$  and the corresponding right relative Cauchy–Green tensor is  $\mathbf{C}_t(s) = \mathbf{F}_t(s)^T \mathbf{F}_t(s)$ .

Attention is restricted to incompressible nonlinear viscoelastic solids and fluids. The solids are assumed to be isotropic in their reference configurations, which are taken as their configurations at  $s = 0$ . This paper is concerned with the transient response of membranes that are assumed to be at rest until  $s = 0$  and then subjected to some loading. For the form of viscoelasticity discussed in this paper, the constitutive equation expresses the stress at the current time  $t$  in terms of the deformation history up to time  $t$ . The constitutive equations for nonlinear viscoelastic response are then the counterparts of the Boltzmann [20] and Volterra [21] theories for linear viscoelasticity and have the general form

$$\boldsymbol{\sigma} = -q\mathbf{I} + \mathbf{h}(\mathbf{B}, t) + \int_0^t \mathbf{g}(\mathbf{B}, \mathbf{C}_t(s), t-s) ds, \quad (1)$$

where  $\boldsymbol{\sigma}$  is the Cauchy stress,  $q$  is an arbitrary scalar arising from the incompressibility constraint,  $\mathbf{I}$  is the identity tensor,  $\mathbf{h}$  is an isotropic function of  $\mathbf{B}$  and  $\mathbf{g}$  is an isotropic function of  $\mathbf{B}$  and  $\mathbf{C}_t(s)$ .

The underlying physical motivation is omitted from the following presentation of the models encompassed by Eq. (1). Further details can be found in the references and the review articles by Morman Jr. [22] and Drapaca, Sivaloganathan and Tenti [23].

### 2.1. K-BKZ Fluids

Kaye [24] and Bernstein, Kearsley, and Zapas [25] proposed a model of the form

$$\boldsymbol{\sigma} = -q\mathbf{I} + \int_{-\infty}^t \left\{ \frac{\partial U(I_1, I_2, t-s)}{\partial I_1} \mathbf{C}_t(s)^{-1} - \frac{\partial U(I_1, I_2, t-s)}{\partial I_1} \mathbf{C}_t(s) \right\} ds, \quad (2)$$

in which  $U(I_1, I_2, s)$  is a material property that depends on time and the scalar invariants

$$I_1 = \text{tr}(\mathbf{C}_t(s)^{-1}) \quad I_2 = \text{tr}(\mathbf{C}_t(s)), \quad (3)$$

and  $\lim_{t \rightarrow \infty} U(I_1, I_2, t) = 0$ . On taking into account that the material has been at rest for  $s \in (-\infty, 0]$ , and observing that  $\mathbf{F}_t(0) = \mathbf{F}^{-1}$ , Eq. (2) can be reduced to the form in Eq. (1).

### 2.2. Pipkin–Rogers model

Pipkin and Rogers [26] introduced the constitutive equation

$$\boldsymbol{\sigma} = -q\mathbf{I} + \mathbf{F} \left\{ \mathbf{G}(\mathbf{C}(t), 0) + \int_0^t \frac{\partial \mathbf{G}(\mathbf{C}(s), t-s)}{\partial (t-s)} ds \right\} \mathbf{F}^T, \quad (4)$$

where

$$\mathbf{G}(\mathbf{C}, s) = \phi_0 \mathbf{I} + \phi_1 \mathbf{C} + \phi_2 \mathbf{C}^2, \quad (5)$$

and  $\phi_0, \phi_1, \phi_2$  are scalar functions of the invariants of  $\mathbf{C}$  and the variable ‘ $s$ ’. The function  $\mathbf{G}(\mathbf{C}, s)$  is a material property that represents a tensor-valued deformation dependent stress relaxation function. Using the relation  $\mathbf{F}(s) = \mathbf{F}_t(s)\mathbf{F}$ , Eq. (4) can be written in the form of Eq. (1).

### 2.3. Finite linear viscoelasticity

Coleman and Noll [27] derived a constitutive equation of the form

$$s = -q\mathbf{I} + \hat{\mathbf{h}}(\mathbf{B}) + \int_0^t \hat{\mathbf{g}}(\mathbf{B}, t-s)[\mathbf{C}_t(s) - \mathbf{I}] ds, \quad (6)$$

in which  $\hat{\mathbf{h}}$  is an isotropic function of  $\mathbf{B}$  and  $\hat{\mathbf{g}}(\mathbf{B}, s)$  is a fourth order isotropic tensor function of  $\mathbf{B}$  and the scalar  $s$ . Its representation is too complicated to present here. Eq. (6) is often written in the alternate form

$$s = -q\mathbf{I} + \bar{\mathbf{h}}(\mathbf{B}) + \int_0^t \bar{\mathbf{g}}(\mathbf{B}, t-s) \frac{d}{ds} \mathbf{C}_t(s) ds. \quad (7)$$

### 3. Inflation of a circular membrane by lateral pressure

Consider an initially flat thin sheet of a nonlinear, incompressible, isotropic viscoelastic material. The sheet has thickness  $h_0$  and is clamped along a circular boundary of radius  $R_0$ . A uniform time varying pressure  $p(t)$  is applied to one side resulting in an axially symmetric deformation. It is assumed that  $h_0/R_0 \ll 1$  and that the theory of large deformations of membranes (Green and Adkins [28]) can be used to describe the viscoelastic sheet. In this theory, the mid-surface in the initial configuration deforms into an axially symmetric surface at each time  $t$ . The theory is formulated in terms of the values of the stresses and stretches on the mid-surface, their variation through the deformed thickness being neglected. Each material element is assumed to be in a state of plane stress.

A material particle at  $(R, \Theta, 0)$ ,  $0 \leq R \leq R_0$ , on the mid-surface in the initial configuration is at  $(r, \theta, z)$  on the deformed surface at time  $t$ , where

$$\begin{aligned} r &= r(R, t), \\ \theta &= \Theta, \\ z &= z(R, t). \end{aligned} \quad (3.1)$$

As a result of the axisymmetric deformation, the principal directions of stretch and stress are known *a priori* to be in the meridional, circumferential and normal directions on the deformed surface. Denoting the meridional direction as 1, the circumferential direction as 2 and the normal direction as 3, the current principal stretches are given by

$$\begin{aligned} \lambda_1 &= \left[ \left( \frac{\partial r}{\partial R} \right)^2 + \left( \frac{\partial z}{\partial R} \right)^2 \right]^{1/2}, \\ \lambda_2 &= r/R, \\ \lambda_3 &= (\lambda_1 \lambda_2)^{-1}, \end{aligned} \quad (3.2)$$

where the expression for  $\lambda_3$  is a consequence of incompressibility.

The principal stresses in the meridional and circumferential directions of the surface are  $\sigma_1$  and  $\sigma_2$ , respectively. The principal stress normal to the deformed surface is neglected, i.e.  $\sigma_3 \approx 0$ . Expressions for the principal stresses  $\sigma_1$  and  $\sigma_2$  are obtained by using representations for  $\mathbf{h}$  and  $\mathbf{g}$  in Eq. (1), the result that  $\mathbf{B}$  and  $\mathbf{C}_t(s)$  have diagonal matrices with respect to the principle directions and the condition  $\sigma_3 \approx 0$  to determine the indeterminate scalar  $q$ . It is found that

$$\sigma_\alpha(t) = A_\alpha(\lambda(t), t) + \int_0^t B_\alpha[\lambda(t), \lambda(s), t-s] ds, \quad \alpha = 1, 2, \quad (3.3)$$

in which  $\lambda(s)$  denotes the pair  $(\lambda_1(s), \lambda_2(s))$ .

The stress resultants in the surface per unit current length along circumferential and meridional directions are, respectively,

$$\begin{aligned} T_1 &= h_o \lambda_3 \sigma_1, \\ T_2 &= h_o \lambda_3 \sigma_2. \end{aligned} \quad (3.4)$$

The equilibrium equations in the meridional and normal directions on the current configuration are, respectively,

$$\frac{\partial T_1}{\partial r} + \frac{T_1 - T_2}{r} = 0, \quad (3.5)$$

and

$$\kappa_1 T_1 + \kappa_2 T_2 = p, \quad (3.6)$$

where  $p$  is the current pressure and  $\kappa_1$  and  $\kappa_2$  are the principle curvatures of the current deformed surface. Yang and Feng [14] have obtained the following expressions for  $\kappa_1$  and  $\kappa_2$  in terms of  $\lambda_1$ ,  $\lambda_2$  and the auxiliary geometric variable  $\eta = \partial r / \partial R$ ,

$$\begin{aligned} \kappa_1 &= -\frac{1}{\lambda_1 [1 - (\eta/\lambda_1)^2]^{1/2}} \frac{d(\eta/\lambda_1)}{dR}, \\ \kappa_2 &= \frac{[1 - (\eta/\lambda_1)^2]^{1/2}}{R\lambda_2}. \end{aligned} \quad (3.7)$$

According to Eqs. (3.1)–(3.3), the stretch ratios and stresses are functions of  $R$  and  $t$  whereas Eq. (3.5) is expressed in terms of the dependent variable ‘ $r$ ’. Eq. (3.5) can be expressed in terms of the independent variable  $R$  by use of the transformation  $r = r(R, t)$

$$\frac{\partial T_1}{\partial R} + \frac{\eta(T_1 - T_2)}{R\lambda_2} = 0. \quad (3.8)$$

Substituting from Eqs. (3.2)–(3.4) and (3.7) into (3.6) and (3.8) leads to a system of two coupled nonlinear equations for  $r(R, t)$  and  $z(R, t)$  that involve second derivatives with respect to  $R$ . A more convenient formulation has been proposed by Feng and Yang [14] and modified by Wineman [16] that leads to a system of first order equations in  $R$  for the dependent variables  $\lambda_1$ ,  $\lambda_2$  and  $\hat{\eta} = \eta/\lambda_1$ .

Introduce the modified set of variables

$$\hat{\eta} = \eta/\lambda_1, \quad \hat{\sigma}_1 = \sigma_1/\lambda_1, \quad \hat{\sigma}_2 = \sigma_2/\lambda_2. \quad (3.9)$$

Note that eliminating the current radius between  $\lambda_2$  and  $\eta$  leads to the relation

$$\frac{\partial \lambda_2}{\partial R} = \frac{\lambda_1 \hat{\eta} - \lambda_2}{R}. \quad (3.10)$$

In view of (3.9), Eq. (3.3) can be restated as

$$\hat{\sigma}_\alpha(t) = \hat{A}_\alpha(\lambda(t), t) + \int_0^t \hat{B}_\alpha[\lambda(t), \lambda(s), t-s] ds, \quad \alpha = 1, 2. \quad (3.11)$$

Eq. (3.8), along with Eqs. (3.9) and (3.10), becomes

$$\frac{\partial \hat{\sigma}_1}{\partial R} + \frac{\hat{\sigma}_1 - \hat{\eta} \hat{\sigma}_2}{R} = 0. \quad (3.12)$$

Let Eq. (3.11) be substituted into Eq. (3.12) and use be made of Eq. (3.10),

$$\begin{aligned}
& \frac{\partial \lambda_1(t)}{\partial R} \left\{ F_1[\lambda(t), t] + \int_0^t G_1[\lambda(t), \lambda(s), t-s] ds \right\} \\
& + \int_0^t \frac{\partial \lambda_1(s)}{\partial R} G_2[\lambda(t), \lambda(s), t-s] ds \\
& = \frac{1}{R} \left\{ F_2[\lambda(t), \hat{\eta}(t), t] + \int_0^t G_3[\lambda(t), \lambda(s), \hat{\eta}(t), \hat{\eta}(s), t-s] ds \right\}.
\end{aligned} \quad (3.13)$$

It is straightforward to calculate expressions for  $F_1, F_2, G_1, G_2, G_3$  in terms of  $\hat{A}_\alpha$  and  $\hat{B}_\alpha$ , but these are omitted for the sake of brevity. Eq. (3.6) along with Eqs. (3.4), (3.7) and (3.9) leads to

$$\frac{\partial \hat{\eta}}{\partial R} = \frac{[1 - \hat{\eta}^2]}{R} \frac{\hat{\sigma}_2}{\hat{\sigma}_1} - \frac{p}{h_o} \frac{\lambda_1 \lambda_2 [1 - \hat{\eta}^2]^{1/2}}{\hat{\sigma}_1}. \quad (3.14)$$

Eqs. (3.10), (3.13) and (3.14) with (3.11) form a system of three equations for  $\lambda_1, \lambda_2$  and  $\hat{\eta}$ . The boundary conditions are (for details, see Wineman [16]):

$$\begin{aligned}
\lambda_1(0, t) &= \lambda_2(0, t), & \hat{\eta}(0, t) &= 1, \\
\lambda_2(R_o, t) &= 1.
\end{aligned} \quad (3.15)$$

This defines a two point boundary value problem for  $\lambda_1, \lambda_2$  and  $\hat{\eta}$  in terms of a system of partial differential–Volterra integral equations.

It is useful to note that the principal curvatures satisfy the Codazzi relation (Green and Adkins [28])

$$\frac{\partial(r\kappa_2)}{\partial R} = \frac{\partial r}{\partial R} \kappa_1. \quad (3.16)$$

Green and Adkins used this relation along with Eqs. (3.5)–(3.7) to obtain the first integral

$$2T_1\kappa_2 = p, \quad 0 \leq R \leq R_o. \quad (3.17)$$

Eq. (3.17) represents the balance along the axis of symmetry of the forces on a ‘cap’ of the inflated membrane from the crown to the edge at radius  $r(R, t)$ ,  $R \leq R_o$ . Eq. (3.17) states that the resultant force from the pressure on the surface of the cap is balanced by the resultant force from the tensile stress along the edge. From this is found

$$\hat{\eta} = \pm \left[ 1 - \left( \frac{pR\lambda_2^2}{2h_o\hat{\sigma}_1} \right)^2 \right]^{1/2}. \quad (3.18)$$

Treloar’s experiments [1] show that an elastic membrane can bulge over its clamped support. This corresponds to  $\eta = \partial r / \partial R$  changing sign at some interior radius  $R' < R_o$ . In other words, the plus sign applies for  $0 \leq R \leq R'$  and the minus sign applies for  $R' \leq R \leq R_o$ . In spite of this difficulty, this relation is useful in the method of numerical solution presented in Section 5.

#### 4. Inflation and extension of a cylindrical tubular membrane

Consider a thin walled tube of a nonlinear, incompressible, isotropic viscoelastic material. In its initial configuration, it has a circular cylindrical mid-surface of radius  $R_o$ , length  $2L_o$  and uniform wall thickness  $h_o$ . The ends are bonded to rigid circular discs of radius  $R_o$ . These are assumed to be part of a loading device that applies a prescribed time dependent internal pressure  $p(t)$  and either prescribed time dependent forces  $F(t)$  normal to the discs or the current length  $2L(t)$ .

Much of the formulation of Section 3 applies here and only the details pertinent to the tubular membrane are given. It is assumed that  $h_o/R_o \ll 1$  and that the theory of large deformation of membranes applies. The deformation is referred to a cylindrical coordinate system whose  $z$ -axis is along the centerline of the tube and whose origin is at the midpoint between the ends. The initially cylindrical mid-surface forms an axially symmetric surface at each time  $t$  whose radius varies along the length. A material particle at  $(R_o, \Theta, Z)$  on the mid-surface in the initial configuration

is at  $(r, \theta, z)$  on the surface in the current configuration, where

$$\begin{aligned} r &= r(Z, t), \\ \theta &= \Theta, \\ z &= z(Z, t). \end{aligned} \quad (4.1)$$

The meridional (1), circumferential (2) and normal directions (3) of the deformed surface are principal directions of stretch and stress. The current principal stretches are given by

$$\begin{aligned} \lambda_1 &= \left[ \left( \frac{\partial r}{\partial Z} \right)^2 + \left( \frac{\partial z}{\partial Z} \right)^2 \right]^{1/2}, \\ \lambda_2 &= r/R_o, \\ \lambda_3 &= (\lambda_1 \lambda_2)^{-1}. \end{aligned} \quad (4.2)$$

The principal stresses in the meridional and circumferential directions of the surface are  $\sigma_1$  and  $\sigma_2$ , respectively. The principal stress normal to the deformed surface is neglected, i.e.  $\sigma_3 \approx 0$ . The principal stresses and stretches are related as in Eq. (3.3).

The stress resultants in the surface per unit current length along circumferential and meridional directions are given by Eq. (3.4). The equilibrium equations in the meridional and normal directions in the current configuration are given by Eqs. (3.5) and (3.6). The principal curvatures  $\kappa_1$  and  $\kappa_2$  are given in terms of  $\lambda_1$ ,  $\lambda_2$  and an associated geometric variable  $\eta = \partial r / \partial Z$  by

$$\begin{aligned} \kappa_1 &= -\frac{1}{\lambda_1 [1 - (\eta/\lambda_1)^2]^{1/2}} \frac{d(\eta/\lambda_1)}{dZ}, \\ \kappa_2 &= \frac{[1 - (\eta/\lambda_1)^2]^{1/2}}{R_o \lambda_2}. \end{aligned} \quad (4.3)$$

Eq. (3.5) can be expressed in terms of the independent variable  $Z$  by use of the transformation  $r = r(Z, t)$ ,

$$\frac{\partial T_1}{\partial Z} + \frac{\eta(T_1 - T_2)}{R_o \lambda_2} = 0. \quad (4.4)$$

A system of equations is now established for the dependent variables  $\lambda_1$ ,  $\lambda_2$  and  $\hat{\eta} = \eta/\lambda_1$ . Recall the modified set of variables  $\hat{\eta}$ ,  $\hat{\sigma}_1$ ,  $\hat{\sigma}_2$  introduced in Eq. (3.9). Expressions for  $\hat{\sigma}_1$  and  $\hat{\sigma}_2$  are given by Eq. (3.11). Note that eliminating the current radius between  $\lambda_2$  and  $\eta$  leads to the relation

$$\frac{\partial \lambda_2}{\partial Z} = \frac{\lambda_1 \hat{\eta}}{R_o}. \quad (4.5)$$

On making use of Eqs. (3.9) and (4.5), Eq. (4.4) becomes

$$\frac{\partial \hat{\sigma}_1}{\partial Z} = \frac{\hat{\eta} \hat{\sigma}_2}{R_o}. \quad (4.6)$$

If the expression for  $\hat{\sigma}_1$  in Eq. (3.11) is substituted into Eq. (4.6) and use is made of Eq. (4.5), the result is

$$\begin{aligned} \frac{\partial \lambda_1(t)}{\partial Z} \left\{ F_1[\lambda(t), t] + \int_0^t G_1[\lambda(t), \lambda(s), t-s] ds \right\} + \int_0^t \frac{\partial \lambda_1(s)}{\partial Z} G_2[\lambda(t), \lambda(s), t-s] ds \\ = \frac{1}{R_o} \left\{ F_2[\lambda(t), \hat{\eta}(t), t] + \int_0^t G_3[\lambda(t), \lambda(s), \hat{\eta}(t), \hat{\eta}(s), t-s] ds \right\}. \end{aligned} \quad (4.7)$$

Eq. (3.6), with Eqs. (3.4), (3.9) and (4.3), becomes

$$\frac{\partial \hat{\eta}}{\partial Z} = \frac{[1 - \hat{\eta}^2] \hat{\sigma}_2}{R_o \hat{\sigma}_1} - \frac{p}{h_o} \frac{\lambda_1 \lambda_2 [1 - \hat{\eta}^2]^{1/2}}{\hat{\sigma}_1}. \quad (4.8)$$

Eqs. (4.5), (4.7) and (4.8) with (3.11) form a system of three equations for  $\lambda_1$ ,  $\lambda_2$  and  $\hat{\eta}$ .

Before discussing boundary conditions, it is important to note that the Codazzi relation in Eq. (3.16) and the equilibrium equations lead to

$$2\pi r^2 \kappa_2 T_1 = p\pi r^2 + F. \quad (4.9)$$

This holds for  $0 \leq Z \leq L_o$  and each time  $t$ . Eq. (4.9) represents the balance along the axis of symmetry of the forces on a ‘cap’ of the tube-disc assembly from the end disc to an edge at  $z(Z, t)$ ,  $Z \leq L_o$ . In particular, the resultant force from the pressure on the interior surface of the tube-disc cap and the external force on the end disc is balanced by the resultant force from the tensile stress along the edge.

Two boundary conditions are

$$\begin{aligned} \hat{\eta}(0, t) &= 0, \\ \lambda_2(L_o, t) &= 1. \end{aligned} \quad (4.10)$$

If  $L(t)$  is prescribed, then the first of Eq. (4.2) implies

$$z(L_o, t) - z(0, t) = L(t) = \int_0^{L_o} \lambda_1 [1 - \hat{\eta}^2]^{1/2} dZ \quad (4.11)$$

at each time  $t$ . If  $F(t)$  is prescribed, Eq. (4.9) requires

$$F = \left\{ 2\pi r(h_o \lambda_3 \sigma_1) - p\pi r^2 \right\}_{Z=0} \quad (4.12)$$

at each time  $t$ . This formulation again leads to a two point boundary value problem for  $\lambda_1$ ,  $\lambda_2$  and  $\hat{\eta}$  that involves a system of partial differential–Volterra integral equations.

This section concludes with the observation that Eq. (4.9) implies

$$\hat{\eta} = - \left[ 1 - \left( \frac{F + p\pi R_o^2 \lambda_2^2}{2\pi h_o \hat{\sigma}_1} \right)^2 \right]^{1/2}. \quad (4.13)$$

## 5. Numerical method of solution

The systems of equations for  $\lambda_1$ ,  $\lambda_2$  and  $\hat{\eta}$  formulated in Sections 3 and 4 have two sources of nonlinearity, large deformations and nonlinear material response. The mathematical forms of the material properties in the constitutive equations presented in Section 2 introduce nonlinearity through their representation of experimental data. Moreover, the governing equations combine features of nonlinear ordinary differential equations that determine the spatial variation as in nonlinear elasticity and of Volterra integral equations that determine time evolution as in viscoelasticity. Thus, in general, solutions to the boundary problems cannot be obtained by analytical methods but, instead, by numerical methods. The method outlined here combines the numerical solution of nonlinear ordinary differential equations and Volterra integral equations.

Consider first the solution at time  $t = 0$  when the integrals vanish in Eqs. (3.11), (3.13) and (4.7). Eqs. (3.10), (3.11), (3.13) and (3.14) for the circular membrane problem can be written in the form

$$\frac{d\Lambda_o^{\text{circ}}}{dR} = \Phi_o^{\text{circ}}(\Lambda_o^{\text{circ}}, R), \quad \Lambda_o^{\text{circ}} = \begin{Bmatrix} \lambda_1(R, 0) \\ \lambda_2(R, 0) \\ \hat{\eta}(R, 0) \end{Bmatrix}. \quad (5.1)$$

Eq. (5.1), subject to the boundary conditions in Eq. (3.15), define a two point boundary value problem. The boundary condition  $\lambda_o = \lambda_1(0, 0) = \lambda_2(0, 0)$  must be determined so that  $\lambda_2(R_o, 0) = 1$ , i.e. a one dimensional shooting method. Eqs. (4.5), (4.7), (4.8) and (3.11) for the tubular membrane problem can be written in the form

$$\frac{d\Lambda_o^{\text{tube}}}{dZ} = \Phi_o^{\text{tube}}(\Lambda_o^{\text{tube}}), \quad \Lambda_o^{\text{tube}} = \begin{Bmatrix} \lambda_1(Z, 0) \\ \lambda_2(Z, 0) \\ \hat{\eta}(Z, 0) \end{Bmatrix}. \quad (5.2)$$



Eq. (5.2), subject to the boundary conditions in Eq. (4.10) and either Eq. (4.11) or (4.12), defines another two point boundary value method. This is a two dimensional shooting problem where boundary conditions  $\lambda_1(0, 0)$  and  $\lambda_2(0, 0)$  must be determined so that  $\lambda_2(L_0, 0) = 1$  and either Eq. (4.11) or (4.12) is satisfied.

Now consider times  $t > 0$ . Let  $t_i, i = 1, \dots, N_{\max}$ , be a set of times at which the solution is to be computed, with  $t_1 = 0$ . Note that the left hand side of Eq. (3.13) (or Eq. (4.7)) is a linear Volterra integral operator on  $\partial\lambda_1/\partial R$  (or  $\partial\lambda_1/\partial Z$ ). Following the approach introduced in linear viscoelasticity by Lee and Rogers [29], applied by Wineman [16,19] to nonlinear viscoelastic membrane examples and discussed by Linz [30], consider Eq. (3.13) at  $t = t_n > t_1$ . Let the integrals have finite sum approximations on a set of times  $t_i, i = 1, \dots, n \leq N_{\max}$ ,

$$\begin{aligned} \int_0^t \frac{\partial\lambda_1(s)}{\partial R} G_2[\lambda(t), \lambda(s), t-s] ds &= \int_{t_1}^{t_n} \frac{\partial\lambda_1(s)}{\partial R} G_2[\lambda(t_n), \lambda(s), t_n-s] ds \\ &\approx W_{nn}^{(2)} \frac{\partial\lambda_1(t_n)}{\partial R} G_2[\lambda(t_n), \lambda(t_n), 0] \\ &\quad + \sum_{k=2}^{n-1} W_{nk}^{(2)} \frac{\partial\lambda_1(t_k)}{\partial R} G_2[\lambda(t_n), \lambda(t_k), t_n-t_k] \end{aligned} \quad (5.3a)$$

$$\begin{aligned} \int_0^t G_1[\lambda(t), \lambda(s), t-s] ds &= \int_{t_1}^{t_n} G_1[\lambda(t_n), \lambda(s), t_n-s] ds \\ &\approx W_{nn}^{(1)} G_1[\lambda(t_n), \lambda(t_n), 0] + \sum_{k=1}^{n-1} W_{nk}^{(1)} G_1[\lambda(t_n), \lambda(t_k), t_n-t_k] \end{aligned} \quad (5.3b)$$

$$\begin{aligned} \int_0^t G_3[\lambda(t), \lambda(s), \hat{\eta}(t), \hat{\eta}(s), t-s] ds &= \int_{t_1}^{t_n} G_3[\lambda(t_n), \lambda(s), \hat{\eta}(t_n), \hat{\eta}(s), t_n-s] ds \\ &\approx W_{nn}^{(3)} G_3[\lambda(t_n), \lambda(t_n), \hat{\eta}(t_n), \hat{\eta}(t_n), 0] \\ &\quad + \sum_{k=1}^{n-1} W_{nk}^{(3)} G_3[\lambda(t_n), \lambda(t_k), \hat{\eta}(t_n), \hat{\eta}(t_k), t_n-t_k]. \end{aligned} \quad (5.3c)$$

The coefficients  $W_{nk}^{(\alpha)}$  depend on the particular method used to approximate the integrals and the time intervals.

Next, let the integrals in Eq. (3.13) be replaced by their approximations in Eq. (5.3). The result is a linear equation for  $\partial\lambda_1(t_n)/\partial R$  whose solution can be written as

$$\frac{\partial\lambda_1(t_n)}{\partial R} = \Phi_n \left[ \lambda(t_n), \hat{\eta}(t_n), R; \left( \lambda_1(t_k), \hat{\eta}(t_k), \frac{\partial\lambda_1(t_k)}{\partial R} \right) \Big|_{k=1}^{n-1} \right], \quad (5.4)$$

where the expression following the semi-colon indicates dependence on the indicated variables at all times from  $t_1$  to  $t_{n-1}$ . Suppose  $\lambda(R, t_k)$ ,  $\hat{\eta}(R, t_k)$  and  $\partial\lambda_1(R, t_k)/\partial R$  have been found for  $t_k < t_n$ . These can be regarded as known functions of  $R$  and Eq. (5.4) can be re-expressed as

$$\frac{\partial\lambda_1(t_n)}{\partial R} = \Phi_n^{(1)} [\lambda(t_n), \hat{\eta}(t_n), R]. \quad (5.5)$$

Eq. (3.10), evaluated at  $t = t_n$ , can be written in a manner similar to Eq. (5.5),

$$\frac{\partial\lambda_2(t_n)}{\partial R} = \Phi_n^{(2)} [\lambda(t_n), \hat{\eta}(t_n), R]. \quad (5.6)$$

If the integrals in Eq. (3.9) are also approximated as in Eq. (5.3), then

$$\hat{\sigma}_\alpha(t_n) = \sigma'_\alpha(\lambda(t_n), R), \quad (5.7)$$

and Eq. (3.14) is also of the form

$$\frac{\partial\hat{\eta}(t_n)}{\partial R} = \Phi_n^{(3)} [\lambda(t_n), \hat{\eta}(t_n), R]. \quad (5.8)$$

The system of Eqs. (5.5), (5.6) and (5.8) can be stated in a manner similar to that in Eq. (5.1)

$$\frac{d\Lambda_n^{\text{circ}}}{dR} = \Phi_n^{\text{circ}}(\Lambda_n^{\text{circ}}, R), \quad \Lambda_n^{\text{circ}} = \begin{Bmatrix} \lambda_1(R, t_n) \\ \lambda_2(R, t_n) \\ \hat{\eta}(R, t_n) \end{Bmatrix}. \quad (5.9)$$

A similar discussion for the tubular membrane problem leads to a system of the form

$$\frac{d\Lambda_n^{\text{tube}}}{dZ} = \Phi_n^{\text{tube}}(\Lambda_n^{\text{tube}}, Z), \quad \Lambda_n^{\text{tube}} = \begin{Bmatrix} \lambda_1(Z, t_n) \\ \lambda_2(Z, t_n) \\ \hat{\eta}(Z, t_n) \end{Bmatrix}. \quad (5.10)$$

Eq. (5.9), subject to boundary conditions in Eq. (3.15) at  $t = t_n$ , defines a two point boundary value problem that involves a one dimensional shooting method where the  $\lambda_n = \lambda_1(0, t_n) = \lambda_2(0, t_n)$  is determined to satisfy  $\lambda_2(R_o, t_n) = 1$ . Similarly, Eq. (5.10), subject to the boundary conditions in Eq. (4.10), defines a two point boundary value problem that involves a two dimensional shooting method where boundary conditions  $\lambda_1(0, t_n)$  and  $\lambda_2(0, t_n)$  are determined to satisfy  $\lambda_2(L_o, t_n) = 1$  and either Eq. (4.11) or (4.12).

Once  $\lambda_1$ ,  $\lambda_2$  and  $\hat{\eta}$  (or their numerical approximations) have been found, the mapping in Eq. (3.1) for the circular membrane problem can be found using Eq. (3.2),

$$\begin{aligned} r(R, t) &= R\lambda_2(R, t), \\ z(R, t) &= \int_R^{R_o} \lambda_1 \left[ 1 - \hat{\eta}^2 \right]^{1/2} dR, \end{aligned} \quad (5.11)$$

and the stresses can be found from Eq. (3.3). A similar procedure gives the mapping stresses for the tubular membrane problem.

Only the outline of the numerical method of solution has been given here. There are several practical issues that must be considered in implementing such a method, such as: (a) the method of approximating the integrals, i.e. the trapezoidal rule or Simpson's rule, (b) the times  $t_i$ ,  $i = 1, \dots, N_{\max}$  at which the solution is to be computed, (c) the use of equal or unequal time increments, (d) the numerical method for integrating the system of differential equations, (e) numerical implementation of the shooting method, and (f) avoiding the recalculation of integrals during iteration.

These issues have been addressed and the method has been successfully used in the solution of a variety of problems for a number of material models. Wineman discussed circular membranes of a vulcanized styrene-butadiene rubber [16] and of a K-BKZ fluid (polyisobutylene) [31], Feng [17] discussed some numerical issues in the case of a circular membrane of latex rubber, and Wineman [19] addressed the two dimensional shooting problem for a tubular membrane of K-BKZ fluid. The method has also been used to study the torsion of a viscoelastic slab by Dai, Rajagopal and Wineman [32] and the circular shear of a cylinder of compressible nonlinear viscoelastic material by Waldron Jr. and Wineman [33].

## 6. Discontinuous solutions, branching of solutions

The equations for the response of nonlinear viscoelastic membranes reduce to those for nonlinear elastic membranes when the integral vanishes in Eq. (1) and  $\mathbf{h}(\mathbf{B}, t)$  becomes independent of  $t$ . Consequently, the response of viscoelastic membranes is influenced by the response of elastic membranes. A particular feature that has important implications for viscoelastic response is described here for the inflation of elastic circular membranes. The plot of the inflation pressure  $p$  vs. the maximum inflated height  $z(0)$  may not be monotonic. It may have an up–down–up shape with a local maximum  $p_1$  at  $z_1$  and a local minimum  $p_2 < p_1$  at  $z_2 > z_1$ . The pressure increases for  $0 \leq z(0) \leq z_1$ , decreases for  $z_1 \leq z(0) \leq z_2$  and increases for  $z_2 < z(0)$ . When  $z(0)$  is specified, a unique value can be found for the pressure. On the other hand, as the specified pressure increases from zero, there is a unique value of  $z(0)$  when  $p < p_2$ , three possible values when  $p_2 < p < p_1$  and a unique value when  $p_1 < p$ . Whether or not the plot of  $p$  vs.  $z(0)$  has an up–down–up shape depends on the material properties. For a specific elastic material model, the shape of the plot depends on the range of values of the material parameters (see Beatty [13]). This behavior is observed for many of the elastic membrane examples mentioned in Section 1.

The consequences of this response are illustrated for the inflation of a nonlinear viscoelastic spherical membrane of a K-BKZ fluid. This geometry allows the essential features of the response to be discussed without introducing the complication of a spatially inhomogeneous deformation. In its initial configuration, the thickness of the membrane is  $h_o$  and the radius of the mid-surface is  $R_o$ . The membrane stays spherical as it expands under an internal time varying pressure  $p(t)$ , with the current radius of the mid-surface being denoted by  $r(t)$ . The stretch ratios in the meridional and circumferential directions are equal and uniformly distributed. That is, the material undergoes local equal biaxial stretch histories,  $\lambda_1(t) = \lambda_2(t) = \lambda(t)$  where  $\lambda(t) = r(t)/R_o$ . The current thickness is  $h(t) = h_o\lambda_3(t)$ . As a result of incompressibility,  $\lambda_3(t) = \lambda^{-2}$ . Note that  $\lambda(t)$  now represents the common value of the stretch ratios in the meridional and circumferential directions and not the set of the stretch ratios as in Sections 3 and 4. The stretch histories have the form

$$\begin{aligned}\lambda(s) &= 1, \quad s \in (-\infty, 0), \\ \lambda(s) &\dots \text{arbitrary}, \quad s \geq 0.\end{aligned}\tag{6.1}$$

Eq. (3.6), in the direction normal to the membrane, is the only non-trivial equilibrium equation. Because of symmetry,  $\kappa_1 = \kappa_2 = 1/r$ ,  $\sigma_1 = \sigma_2 = \sigma$  and Eq. (3.6) reduces to

$$2\frac{h}{r}\sigma = p.\tag{6.2}$$

For equal biaxial stretch histories as in Eq. (6.1), the K-BKZ constitutive equation in Eqs. (2) and (3) becomes

$$\sigma(t) = \int_t^\infty \left( \lambda^2(t) - \frac{1}{\lambda^4(t)} \right) M(\lambda(t), s) ds + \int_0^t \left( \frac{\lambda^2(t)}{\lambda^2(s)} - \frac{\lambda^4(s)}{\lambda^4(t)} \right) M\left(\frac{\lambda(t)}{\lambda(s)}, t-s\right) ds,\tag{6.3a}$$

where

$$M(\lambda, s) = \frac{\partial U(I_1(\lambda), I_2(\lambda), s)}{\partial I_1} + \lambda^2 \frac{\partial U(I_1(\lambda), I_2(\lambda), s)}{\partial I_2},\tag{6.3b}$$

and

$$I_1(\lambda) = 2\lambda^2 + \frac{1}{\lambda^4} \quad I_2(\lambda) = \lambda^4 + \frac{2}{\lambda^2}.\tag{6.3c}$$

For convenience, attention is restricted to the special case in which the material property  $U(I_1, I_2, t)$  has the form

$$U = -\frac{dG(t)}{dt} \hat{U}(I_1, I_2),\tag{6.4}$$

in which  $G(t)$ , a stress relaxation function, monotonically decreases and  $G(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Eq. (6.3a) reduces to

$$\sigma(t) = \left( \lambda^2(t) - \frac{1}{\lambda^4(t)} \right) \hat{M}(\lambda(t)) G(t) - \int_0^t \left( \frac{\lambda^2(t)}{\lambda^2(s)} - \frac{\lambda^4(s)}{\lambda^4(t)} \right) \hat{M}\left(\frac{\lambda(t)}{\lambda(s)}\right) \dot{G}(t-s) ds,\tag{6.5}$$

in which

$$\hat{M}(\lambda) = \frac{\partial \hat{U}(I_1(\lambda), I_2(\lambda))}{\partial I_1} + \lambda^2 \frac{\partial \hat{U}(I_1(\lambda), I_2(\lambda))}{\partial I_2}.\tag{6.6}$$

Using the expressions for  $h$  and  $\lambda_3$  in terms of  $\lambda$ , Eq. (6.2) becomes

$$\frac{p(t)R_o}{2h_o} = \frac{\sigma(t)}{\lambda^3(t)}.\tag{6.7}$$

Combining Eqs. (6.5) and (6.7) gives

$$\frac{p(t)R_o}{2h_o} = \frac{1}{\lambda^3(t)} \left\{ \left( \lambda^2(t) - \frac{1}{\lambda^4(t)} \right) \hat{M}(\lambda(t)) G(t) - \int_0^t \left( \frac{\lambda^2(t)}{\lambda^2(s)} - \frac{\lambda^4(s)}{\lambda^4(t)} \right) \hat{M}\left(\frac{\lambda(t)}{\lambda(s)}\right) \dot{G}(t-s) ds \right\}.\tag{6.8}$$

For a specified pressure history this is a nonlinear Volterra integral equation for the stretch ratio history. As is to be shown, Eq. (6.8) has properties that are similar to those in the equation for an inflated elastic spherical membrane and that influence the evolution of the stretch history. Thus, assume that the inflation history up to the current time,  $\lambda(s)$ ,  $0 \leq s < t$ , is known and restate Eq. (6.8) as

$$\frac{p(t)R_o}{2h_o} = f[x; \lambda(s)|_0^t] \quad (6.9)$$

where

$$f[x; \lambda(s)|_0^t] = \frac{1}{x^3} \left\{ \left( x^2 - \frac{1}{x^4} \right) \hat{M}(x)G(t) - \int_0^t \left( \frac{x^2}{\lambda^2(s)} - \frac{\lambda^4(s)}{x^4} \right) \hat{M}\left(\frac{x}{\lambda(s)}\right) \dot{G}(t-s)ds \right\}. \quad (6.10)$$

A solution  $\lambda(t)$  of Eqs. (6.9) and (6.10) satisfies

$$\frac{p(t)R_o}{2h_o} = f[\lambda(t); \lambda(s)|_0^t]. \quad (6.11)$$

Consider Eqs. (6.9) and (6.10) at  $t = 0$ ,

$$\begin{aligned} \frac{p(0)R_o}{2h_o} &= f[x; \lambda(s)|_0^0] \\ &= \frac{1}{x^3} \left( x^2 - \frac{1}{x^4} \right) \hat{M}(x)G(0). \end{aligned} \quad (6.12)$$

Eq. (6.12) is the same as the equation for the inflation of an elastic spherical membrane. Adkins and Rivlin [2] have discussed the right hand side of Eq. (6.12) when  $\hat{M}(x)$  in Eq. (6.6) is given by

$$\hat{M}(\lambda) = C_1 + \lambda^2 C_2, \quad (6.13)$$

where  $C_1$  and  $C_2$  are constants. They showed that the right hand side of Eq. (6.12) has an up–down–up shape when  $C_2/C_1 < 0.214$ . Now let  $\hat{M}(x)$  be any function of  $x$  such that the plot of the right hand side of Eq. (6.12) with respect to  $x$  has an up–down–up shape. The  $x$  dependence of  $f[x; \lambda(s)|_0^t]$  when  $t > 0$  varies with the evolution of the history  $\lambda(s)$ ,  $0 \leq s < t$ , but the up–down–up shape may persist when  $t \in [0, t_o]$  for some time  $t_o > 0$ .

In order to see the implications of this up–down–up shape, let Eq. (6.11) be differentiated with respect to time,

$$\frac{d}{dt} \left[ \frac{p(t)R}{2h_o} \right] = f_1[\lambda(t); \lambda(s)|_0^t] \frac{d\lambda(t)}{dt} + f_2[\lambda(t); \lambda(s)|_0^t]. \quad (6.14a)$$

The expression

$$f_1[\lambda(t); \lambda(s)|_0^t] = \frac{\partial}{\partial x} f[x; \lambda(s)|_0^t]_{x=\lambda(t)}, \quad (6.14b)$$

represents the slope of  $f[x; \lambda(s)|_0^t]$  at  $x$ . The expression

$$f_2[\lambda(t); \lambda(s)|_0^t] = \frac{\partial}{\partial t} f[x; \lambda(s)|_0^t]_{x=\lambda(t)} \quad (6.14c)$$

accounts for change in  $f[x; \lambda(s)|_0^t]$  as the history evolves. From (6.14a), it is found

$$\frac{d\lambda(t)}{dt} = \frac{\frac{d}{dt} \left[ \frac{p(t)R_o}{2h_o} \right] - f_2[\lambda(t); \lambda(s)|_0^t]}{f_1[\lambda(t); \lambda(s)|_0^t]}. \quad (6.15)$$

Consider time  $t^*$ , when  $f[x; \lambda(s)|_0^t]$  has a local maximum at  $x^*$ . Suppose the solution to Eq. (6.9) at time  $t^*$  coincides with the local maximum. Then  $\lambda(t^*) = x^*$  satisfies

$$\frac{p(t^*)R_o}{2h_o} = f[\lambda(t^*); \lambda(s)|_0^{t^*}]. \quad (6.16)$$

According to Eq. (6.15)

$$f_1 \left[ \lambda(t^*); \lambda(s) \big|_0^{t^*} \right] = \frac{\partial}{\partial x} f \left[ x; \lambda(s) \big|_0^{t^*} \right]_{x=\lambda(t^*)} = 0,$$

and the solution is bounded but has an infinite slope at  $t = t^*$ . There are various possibilities for the solution at time  $t^*+$ . It may lie on the rising portion of  $f[x; \lambda(s) \big|_0^{t^*+}]$  after its upturn following the local minimum and may therefore have a jump discontinuity. The solution can then be described as expanding out of control much like the rapid growth of a bubble gum bubble. Alternatively, there may be solutions on either side of the local maximum, in which case the solution develops two branches. On the other hand, the up–down–up shape may not persist. There may be a time  $t_1 > 0$  such that  $f[x; \lambda(s) \big|_0^{t_1}]$  becomes monotonic in  $x$  for  $t > t_1$ . In this case there is a unique continuous solution. Wineman [34] has exhibited solutions when all of these possibilities occur.

This behavior has been shown to occur in the elongation and inflation of a tubular membrane in [19]. The stretch ratios in both the meridional and circumferential directions at the midpoint of the membrane approach infinite slopes at a finite time. This appears to be the only example of such behavior for nonlinear viscoelastic membranes undergoing inhomogeneous deformations.

## 7. Concluding comments

The subject of nonlinear viscoelastic membranes is an important one within continuum mechanics that combines nonlinear elasticity with Volterra integral equations. The examples in Sections 3 and 4 show that a nonlinear elastic membrane problem has a corresponding nonlinear viscoelastic membrane problem. In other words, there is a correspondence ‘approach’ in the nonlinear theories that is analogous to the correspondence principle connecting linear elasticity and viscoelasticity. However, the methods differ for using an elasticity solution to obtain a viscoelasticity solution. The latter case requires the development of new computational methods. As shown in Section 6, when the phenomena exhibited by nonlinear elastic membranes are embedded in the time dependent response of nonlinear viscoelastic membranes, there can be many interesting mathematical problems. In summary, nonlinear viscoelastic membranes leads to challenging computational and applied mathematics problems.

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